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Approximation Orders of Principal Shift-Invariant Spaces Generated by Box Splines

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We investigate the approximation orders of principal shift-invariant subspaces of $L_p(\mathbb{R}^d)$, $1 < p < \infty$, generated by exponential box splines M associated to rational matrices. Moreover, under some regularity assumptions on M the exact approximation orders are determined.

1. INTRODUCTION

A space of functions S is said to be **shift-invariant** if it is invariant under all integer translates, termed as **shifts** hereafter, that is,

$$f \in S \Leftrightarrow f(\cdot + \alpha) \in S \qquad \text{for all} \quad \alpha \in \mathbb{Z}^d.$$

To construct a shift-invariant (**SI**) subspace of $L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, one usually starts with a finite collection of functions $\Phi := \{\phi_1, ..., \phi_n\}$ and defines S to be the closure, with regards to the topology of $L_p(\mathbb{R}^d)$ (or some other weaker topology), of the span of the shifts of ϕ_i, $i = 1, ..., n$. The simplest type of **SI** space is the so called **principal shift-invariant** space (**PSI** hereafter), $S := S(\phi)$, generated by a singleton function ϕ. In practice, schemes of approximation from **SI** spaces are realized via a family of spaces S^h, $h > 0$, invariant under $h\mathbb{Z}^d$ translates. In most cases, the family S^h, $h > 0$, is generated from S by means of dilation

$$S^h = \{f(\cdot/h) : f \in S\}, \qquad h > 0.$$

In the literature these are known as **stationary** refinements.

The question arises as to whether the spaces $\{S^h\}_h$ are suitable for approximation. It is customary, to measure the efficiency of the family

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$\{S^h\}_h$ for approximation by means of approximation orders. We say that $\{S^h\}_h$ provides **approximation order k**, $k>0$, in $L_p(\mathbb{R}^d)$, $1<p<\infty$, if for any function f in the potential space $\mathscr{H}^k_p$, the **error** of approximation of f satisfies

$$E_p(f, S^h) = O(h^k), \tag{1.1}$$

with the unspecified constant in $O(h^k)$ independent of h. Here, the error of approximation of f from S^h is defined by

$$E_p(f, S^h) := E(f, S^h, L_p(\mathbb{R}^d)) := \inf_{s \in S^h} \|f - s\|_{L_p}, \tag{1.2}$$

and $\mathscr{H}^k_p := \mathscr{H}^k(L_p(\mathbb{R}^d))$ is the potential space

$$\mathscr{H}^k(L_p(\mathbb{R}^d)) := \{f : \|f\|_{\mathscr{H}^k_p} := \|((1+|\cdot|^2)^{k/2}\hat{f})^\vee\|_{L_p} < \infty\}.$$

(Recall that for $k \in \mathbb{N}$, $\mathscr{H}^k(L_p(\mathbb{R}^d))$ is the usual Sobolev space.)

By and large the approximation orders of **SI** spaces have been associated with the so called Strang–Fix conditions (**SF**), that first appeared in the fundamental work of Schoenberg [12] in the 1940's. For an appropriately decaying ϕ the Strang–Fix conditions relate the behavior of the Fourier transform $\hat{\phi}$ at $2\pi\mathbb{Z}^d$ with the order of polynomials that can be reproduced locally from the shifts of ϕ.

The transition from the Strang–Fix conditions to the approximation schemes that are actually employed in the literature is via **quasi-interpolants**, a family of linear operators Q_h, $h>0$ that map a given function f into S^h (for references see [3]). The underlying idea for their construction, is the fact that the finite-dimensional space $\Pi_{<k}$ (of polynomials of degree $<k$), which by **SF** can be shown to lie in $\bigcap_h S^h$, provides **locally** good approximants to smooth functions. Nevertheless, this method imposes some a priori restrictions in the eligibility of ϕ, which in some cases appear to be detrimental. Moreover, it is known that in certain pathological cases quasi-interpolants fall short in realizing the actual approximation orders of shift-invariant spaces (see below for more details).

A breakthrough in the study of the approximation properties of **SI** spaces was recently achieved by de Boor, DeVore and Ron in [1] where they completely characterized the approximation orders of **SI** subspaces of $L_2(\mathbb{R}^d)$. Avoiding the polynomial reproducing arguments, they initiated a new error analysis executed exclusively on the frequency domain making heavy use of the geometric structure of $L_2(\mathbb{R}^d)$. A forerunner of [1] can be found in the work of de Boor and Ron [3] (for $p=\infty$).

Extending the results of [1], in [7] we investigated lower bounds of **PSI** spaces in $L_p(\mathbb{R}^d)$, $1<p<\infty$. In particular, we made use of the theory of convolution operators and we obtained results that do not preassume any decay conditions on the generator ϕ. Nevertheless, in [7] we considered only **stationary** refinements.

The motivation for our present work was a recent paper of Ron [10] where he investigates the approximation orders of **PSI** spaces related to **non-stationary** refinements and he specializes his results to box splines.

The current literature, related to the approximation orders of **PSI** spaces generated by polynomial box splines primarily concerns itself with box splines associated to integer matrices Ξ (that is, $\Xi \in \mathbb{Z}^{d \times n}$) (see the recent book [2] for references), with the exemption being [11] where the above questions are considered for general real matrices in the context of L_∞-approximation. Results regarding the characterization of the approximation orders of **SI** spaces generated by exponential box splines $M_{\Xi, \lambda}$ (see below for notations) with $\Xi \in \mathbb{Z}^{d \times n}$ and general λ have been obtained in [4] and [8].

The approximation schemes in the above references are given by generalized quasi-interpolant arguments; as before one has to identify a finite dimensional space $\mathscr{H}$ (of exponential polynomials or polynomials) in $\bigcap_h S_h^h$, that provides locally good approximation to smooth functions. We note however that the spaces S_h^h are generated using **non-stationary** refinements (see below for definitions).

We intend to investigate the approximation properties of **PSI** spaces generated by exponential box splines related to general rational matrices in the different L_p-metrics for $1 < p < \infty$. By using rational matrices one hopes to minimize the support of $M_{\Xi, \lambda}$ while at the same time retain good approximation properties. It is well known that, for non-integral matrices, schemes based on quasi-interpolants fail to attain the actual approximation orders; an instance of an exponential box spline has been constructed in [10] where the aforementioned finite dimensional space $\mathscr{H} = \{0\}$, however the corresponding shift-invariant spaces provide positive approximation orders. The approach we plan to pursue instead, is reminiscent of the work in [7]. This type of questions has also been considered in [3] (for $p = \infty$) and as we have already mentioned by Ron in [10]. However, the results in [10] were restricted to $2 \leqslant p \leqslant \infty$ and lower bounds for the approximation orders were obtained for a smaller, than the potential spaces, class of smooth functions. With our present work we extend the results reported in [10] in both directions, namely we establish lower bounds for the orders of approximation for the range of p, $1 < p < \infty$, and we also capture then full class of potential spaces. Moreover, in a recent paper of Johnson [5] it has been shown that these lower bounds are upper bounds as well, thus the exact approximation orders of these spaces are determined.

After the completion of this work I was informed that Johnson [6] has obtained some new results on the approximation orders of **PSI** subspaces of L_p, $1 \leqslant p \leqslant \infty$. In particular, for the case of **PSI** spaces generated by exponential box-splines he considers splines associated to real matrices. His notion, though, of *approximation order* k deviates from our definition, in that his smoothness spaces are the Besov spaces $B^{\lambda}_{\infty}(L_p)$, $\lambda > k$, a smaller class of functions than the potential spaces $\mathscr{H}^k_p$.

Throughout this paper we shall use standard multi-index notation. For every multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ and $x = (x_1, ..., x^d) \in \mathbb{R}^d$ we define the symbols

$$x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

$$|\alpha| := \alpha_1 + \cdots + \alpha_d$$

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d},$$

while by $|x|$ we denote the Euclidean norm of the vector x, i.e.,

$$|x|^2 := \sum_{i=1}^{d} |x_i|^2.$$

If $x \in \mathbb{R}^d$ we let e_x denote the exponential function with values

$$e_x(y) = e^{ix \cdot y}, \qquad y \in \mathbb{R}^d,$$

where $x \cdot y$ denotes the inner product of the two vectors. The Fourier transform $\hat{f}$ of an integrable function is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\, e_\xi(-x)\, dx,$$

while the inverse Fourier transform is defined by $\check{f}(\xi) := (2\pi)^{-d} \hat{f}(-\xi)$. Duality extends the Fourier transform, and thus its inverse, uniquely from the space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^d$, $\mathscr{S}(\mathbb{R}^d)$, to its topological dual, $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions. We assume that the reader is familiar with the basic properties of the Fourier transform, in particular it is well known that the Fourier transform is a continuous, linear, one-to-one mapping of $\mathscr{S}'(\mathbb{R}^d)$ onto $\mathscr{S}'(\mathbb{R}^d)$, whose inverse is also continuous.

By σ_h, $h > 0$, we denote the operator acting on functions on $\mathbb{R}^d$ by

$$\sigma_h \colon f(\cdot) \to f(\cdot/h).$$

Finally for any $N \in \mathbb{N}$ and sufficiently smooth function f we define

$$\|f\|_N := \sup_{|\alpha| \leqslant N} \sup_{\omega \in \mathbb{R}^d} |\omega|^{|\alpha|}\, |D^\alpha f(\omega)|. \tag{1.3}$$

2. Lower Bounds for the Approximation Orders of PSI Spaces

In this section we are going to obtain lower bounds for the approximation orders of **PSI** spaces which are, to some extent, a generalization to non-stationary refinements of our work in [7]. We shall consider only **PSI** spaces S generated by compactly supported, complex valued functions. In particular, we define $S := S(\phi)$, $\phi \in L_p(\mathbb{R}^d)$ by

$$S := \left\{ s : s = \sum_{\alpha \in \mathbb{Z}^d} c(\alpha)\, \phi(\cdot - \alpha),\ (c(\alpha)) \in l_p(\mathbb{Z}^d) \right\}. \tag{2.1}$$

To avoid any ambiguity in the above definition we note that the partial sums of the series on the right hand side of (2.1) converge in the sense of L_p-convergence, since ϕ is compactly supported.

We start with a family of compactly supported functions $\{\phi_h\}_{h>0} \in L_p(\mathbb{R}^d)$ and generate the shift-invariant spaces $S_h := S(\phi_h)$. By dilation we obtain the scale of spaces

$$S_h^h := \sigma_h S_h := \{ s(\cdot/h) : s \in S(\phi_h) \}, \qquad h > 0.$$

Our aim is to give necessary conditions on $\{\phi_h\}_{h>0}$ so that for every $f \in \mathscr{H}_p^k$, $k > 0$, the error of approximation of f from S_h^h satisfies

$$E_p(f, S_h^h) \leqslant \text{const}\, h^k\, \|f\|_{\mathscr{H}_p^k}, \qquad 1 < p < \infty, \tag{2.2}$$

where $E_p(f, S_h^h)$ is defined by (1.2). If this is the case we say that the family $\{S_h\}_h$ provides order of approximation k. We note that the constant in (2.2) is independent of f and h; as a matter of fact that the constant that we derive depends only on S and p and blows up as p approaches 1 or ∞.

We set out by making some observations that significantly simplify our subsequent analysis.

First we note that a simple change of variables leads to

$$E_p(f, S^h) = h^{d/p} E_p(\sigma_{1/h} f, S(\phi_h)). \tag{2.3}$$

In addition, we will assume that $\widehat{\sigma_{1/h} f}$ is supported in a neighborhood of 0, $\mathscr{B}_\varepsilon$ for any $\varepsilon > 0$, where $\mathscr{B}_\varepsilon$ is the ball centered at 0 of radius ε (we shall fix ε later in the sequel). For this we recall Michlin's multiplier theorem.

Theorem 2.1. *Let $1 < p < \infty$ and let $m(\omega)$ be a complex-valued function on $\mathbb{R}^d$ such that*

$$\|m\|_N < \infty,$$

cf. (1.3), *for some integer* $N > d/2$. *Then,* m *is a bounded multiplier for* $L_p(\mathbb{R}^d)$ *with multiplier norm* $\|m\|_{\mathscr{M}_p} \leqslant \text{const}_p \|m\|_N$, *i.e., for every* $f \in L_p(\mathbb{R}^d)$

$$\|f * \check{m}\|_{L_p} \leqslant \text{const}_p \|m\|_N \|f\|_{L_p}. \tag{2.4}$$

Proof. For a proof see page 321 of [13]. ▮

We note however, that the constant in (2.4) depends on p and blows up as p approaches the endpoints 1 and ∞. For the rest of the paper we drop the suscript in const_p. Also, we point out for later use that if m_1 and m_2 are two bounded multipliers on $L_p(\mathbb{R}^d)$ then one easily deduces from (2.4)

$$\|m_1 m_1\|_{\mathscr{M}_p} \leqslant \text{const} \|m_1\|_N \|m_2\|_N. \tag{2.5}$$

We let $\varepsilon > 0$ and assume that η is a C_0^∞ function such that

$$\eta(x) = \begin{cases} 1, & x \in \mathscr{B}_{1/2} \\ 0, & x \notin \mathscr{B}_1. \end{cases} \tag{2.6}$$

For notational convenience we define

$$\eta_\varepsilon := \sigma_\varepsilon \eta = \eta(\cdot/\varepsilon). \tag{2.7}$$

It follows easily that

$$\begin{aligned} \|((1-\eta_\varepsilon)\widehat{\sigma_{1/h} f})^\vee\|_{L_p} &= \left\| \left(\frac{(1-\eta_\varepsilon(\cdot))}{(h^2+|\cdot|^2)^{k/2}} (h^2+|\cdot|)^{k/2} \widehat{\sigma_{1/h} f})^\vee(\cdot) \right)^\vee \right\|_{L_p} \\ &\leqslant \text{const} \|((h^2+|\cdot|^2)^{k/2} \widehat{\sigma_{1/h} f})^\vee\|_{L_p} \\ &\leqslant \text{const}\, h^{k-d/p} \|f\|_{\mathscr{H}_p^k}. \end{aligned}$$

Here, in the first inequality we used the fact that $(1-\eta_\varepsilon(\cdot))/(h^2+|\cdot|^2)^{k/2}$, is a bounded multiplier for $L_p(\mathbb{R}^d)$ with norm $\|\cdot\|_{\mathscr{M}_p}$, independent of $h \in (0..1]$.

Therefore, writing, $\widehat{\sigma_{1/h} f} = (1-\eta_\varepsilon)\widehat{\sigma_{1/h} f} + \eta_\varepsilon \widehat{\sigma_{1/h} f}$, we find that

$$|E_p(\sigma_{1/h} f, S(\phi_h)) - E_p((\eta_\varepsilon \widehat{\sigma_{1/h} f})^\vee, S(\phi_h))| \leqslant \text{const}\, h^{k-d/p} \|f\|_{\mathscr{H}_p^k}. \tag{2.8}$$

Summarizing, from (2.3) and (2.8) it suffices to approximate $(\eta_\varepsilon \widehat{\sigma_{1/h} f})^\vee$ from $S(\phi_h)$, i.e., to investigate whether

$$h^{d/p} E_p((\eta_\varepsilon \widehat{\sigma_{1/h} f})^\vee, S(\phi_h)) \leqslant \text{const}\, h^k \|f\|_{\mathscr{H}_p^k}. \tag{2.9}$$

For each $h>0$ our approximant $T_h(f)$ we'll be given in the frequency domain by

$$\widehat{T_h(f)} := \tau_h(\sigma_{1/h} f)\widehat{\phi_h}, \tag{2.10}$$

where

$$\tau_h\colon f \to \sum_{\beta \in 2\pi\mathbb{Z}^d} \frac{\eta_\varepsilon(\cdot+\beta)\,\hat{f}(\cdot+\beta)}{\widehat{\phi_h}(\cdot+\beta)}. \tag{2.11}$$

Here, η_ε is given by (2.7) for some fixed ε, $0<\varepsilon<\pi$.

To see that for each $h>0$ (2.10) gives rise to admissible elements in $S(\phi_h)$ we employ the following theorem from [11]:

THEOREM 2.2. *Let ϕ be a complactly supported distribution and g an infinitely differentiable function. Then*

$$\sum_{\alpha \in \mathbb{Z}^d} g(\alpha)\,\phi(\cdot-\alpha) = \sum_{\beta \in 2\pi\mathbb{Z}^d} \phi * (e_\beta\, g), \tag{2.12}$$

where the convergence in the the right hand side of (2.12) *is taken in the distributional sense.*

Moreover, as it was elaborated in [11], in the case where g and all its derivatives are of polynomial growth at ∞, then the convergence in the right hand side of (2.12) can be also taken in the sense of tempered distributions. This last remark justifies the use of the Fourier transform

As a result of (2.12)

$$T_h(f)(\cdot) = \sum_{\alpha \in \mathbb{Z}^d} g_h(\alpha)\,\phi_h(\cdot-\alpha), \qquad \widehat{g_h}(\cdot) = \frac{\widehat{\sigma_{1/h} f}(\cdot)\,\eta_\varepsilon(\cdot)}{\widehat{\phi_h}(\cdot)}.$$

Since ϕ_h, $h>0$, are compactly supported the smoothness of g_h and the polynomial growth of all its derivatives are guaranteed upon assuming that $\widehat{\phi_h}$, $h>0$, do not vanish in $\mathscr{B}_\varepsilon$. Moreover $(g_h(\alpha)) \in l_p(\mathbb{Z}^d)$. To see this we note that $\operatorname{supp}\widehat{g_h} \subset \mathscr{B}_\varepsilon$ while, by Wiener's lemma, $(\eta_\varepsilon/\widehat{\phi_h})^\vee \in L_1(\mathbb{R}^d)$. These two facts along with well known estimates, regarding the Shannon sampling theorem (cf. [14]), guarantee that

$$\|g_h(\alpha)\|_{l_p(\mathbb{Z}^d)} \approx \|g_h\|_{L_p(\mathbb{Z}^d)} \leqslant \text{const}\, \|f(\cdot/h)\|_{L_p} < \infty.$$

The main result of this section is the following theorem:

THEOREM 2.3. *Let ϕ_h, $h>0$, be a family of compactly supported functions in $L_p(\mathbb{R}^d)$ such that $|\widehat{\phi_h}|>\delta>0$ on some h-independent neighborhood of* 0, $\mathscr{B}_\varepsilon$, $0<\varepsilon<\pi$. *If for some k, $h_0>0$ and $0<h<h_0$ the sequence*

$$m_{h,k,p}: 2\pi\mathbb{Z}^d\backslash 0 \ni \beta \to \left\| \frac{\eta_\varepsilon(\cdot)\, \widehat{\phi_h}(\cdot+\beta)}{(h^2+|\cdot|^2)^{k/2}\, \widehat{\phi_h}(\cdot)} \right\|_{\mathscr{M}_p}$$

is in $l_1(2\pi\mathbb{Z}^d\backslash 0)$ with

$$\sup_{0<h<h_0} \|m_{h,k,p}\|_{l_1(2\pi\mathbb{Z}^d\backslash 0)} < \infty \tag{2.13}$$

then, $\{S_h\}_h$ provides order of approximation k in $L_p(\mathbb{R}^d)$.

Proof. We'll prove that (2.9) holds. Let $0<h<h_0$, then

$$\begin{aligned}
&\|(\eta_\varepsilon \widehat{\sigma_{1/h} f})^\vee - T_h(f)\|_{L_p} \\
&\quad = \left\| \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash\{0\}} \left(\frac{\eta_\varepsilon(\cdot+\beta)\, \widehat{\sigma_{1/h} f}(\cdot+\beta)}{\widehat{\phi_h}(\cdot+\beta)}\, \widehat{\phi_h}(\cdot) \right)^\vee \right\|_{L_p} \\
&\quad = \left\| \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash\{0\}} e_{-\beta} \left(\frac{\eta_\varepsilon(\cdot)\, \widehat{\sigma_{1/h} f}(\cdot)}{\widehat{\phi_h}(\cdot)}\, \widehat{\phi_h}(\cdot-\beta) \right)^\vee \right\|_{L_p} \\
&\quad \leqslant \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash\{0\}} \left\| \left(\frac{\eta_\varepsilon(\cdot)\, \widehat{\sigma_{1/h} f}(\cdot)}{\widehat{\phi_h}(\cdot)}\, \widehat{\phi_h}(\cdot-\beta) \right)^\vee \right\|_{L_p} \\
&\quad = \sum_{\beta\in 2\pi\mathbb{Z}^d\backslash\{0\}} \left\| \left(\frac{\eta_\varepsilon(\cdot)(h^2+|\cdot|^2)^{k/2}\, \widehat{\sigma_{1/h} f}(\cdot)}{(h^2+|\cdot|^2)^{k/2}\, \widehat{\phi_h}(\cdot)}\, \widehat{\phi_h}(\cdot-\beta) \right)^\vee \right\|_{L_p} \\
&\quad \leqslant h^{k-d/p} \|f\|_{\mathscr{H}_p^k} \sup_{0<h<h_0} \|m_{h,k,p}\|_{l_1(2\pi\mathbb{Z}^d\backslash 0)} \\
&\quad \leqslant \text{const}\, h^{k-d/p} \|f\|_{\mathscr{H}_p^k}. \quad \blacksquare
\end{aligned}$$

We note that the above theorem would hold even if ϕ_h, $h>0$, were compactly supported distributions. In that case though, we would have to modify our definition of shift-invariant space.

Although (2.13) appears to be a bit complicated to verify, it is not hard to show that it is satisfied in certain cases of exponential box splines with rational directions. This will be the main theme of our next section.

3. Box Spline Spaces

Let $\Xi \in \mathbb{R}^{d \times n}$ be a $d \times n$ matrix with non-zero column vectors $\{\xi \in \Xi\}$. Let also $\lambda := \{\lambda_\xi\}_{\xi \in \Xi}$ be an associated to Ξ row vector of complex numbers. The **exponential box spline** $M := M_{\Xi, \lambda}$ is defined via its Fourier transform by

$$\hat{M}(\omega) := \prod_{\xi \in \Xi} \frac{e^{\lambda_\xi - i\xi \cdot \omega} - 1}{\lambda_\xi - i\xi \cdot \omega}, \qquad \omega \in \mathbb{R}^d. \tag{3.1}$$

It is known that (3.1) gives rise to a compactly supported measure while in the case where rank $\Xi = d$, which we assume for the rest of the paper, M becomes a piecewise-exponential-polynomial function supported on the zonotope

$$Z_\Xi := \left\{ \sum_{\xi \in \Xi} t_\xi \xi : t_\xi \in [0..1] \right\}.$$

Exponential box splines where introduced by Ron in [9] with the **direction** matrix $\Xi \in \mathbb{Z}^{d \times n}$. We note that for $\lambda = 0$, (3.1) defines the usual **polynomial box splines**.

For a fixed exponential box spline M (that is, for fixed Ξ and λ) and $h > 0$ we set

$$M_h := M_{\Xi, h\lambda}.$$

Then we define the **PSI** space $S := S(M)$ as in (2.1) (since M is compactly supported). In a similar fashion $S_h := S(M_h)$. The approximation properties of $\{S_h\}_h$ are characterized in terms of the scale of spaces

$$S_h^h := \{s(\cdot/h) : s \in S_h, \qquad h > 0\}.$$

Note that unlike the stationary case, where ϕ does not vary with h, here each S_h^h is generated by dilating a different **PSI** space S_h. From (3.1) is apparent that for $\lambda = 0$ the refinements S_h^h become stationary since in this case $M_h = M$, for every $h > 0$.

Before we proceed with the establishment of lower bounds for the approximation orders of $\{S_h\}_h$ we need some further notation.

For every $\beta \in 2\pi\mathbb{Z}^d$ we define

$$K_\beta := \{\xi \in \Xi : \xi \cdot \beta \in 2\pi\mathbb{Z} \backslash 0\}$$

and two disjoint sets, complementary to K_β,

$$L_\beta := \{\xi \in \Xi : \xi \cdot \beta \notin 2\pi\mathbb{Z}\},$$

and

$$O_\beta := \{\xi \in \Xi : \xi \cdot \beta = 0\}.$$

Finally we define

$$k(\Xi) := \min\{\#K_\beta : \beta \in 2\pi\mathbb{Z}^d\}. \tag{3.2}$$

4. Main Results

THEOREM 4.1. *Let* $M = M_{\Xi, \lambda}$, $\Xi \in \mathbb{Q}^{d \times n}$, *be an exponential box spline and let* $k := k(\Xi)$. *If*

$$\sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} \prod_{\xi \in K_\beta \cup L_\beta} |\xi \cdot \beta|^{-1} < \infty, \tag{4.1}$$

then, $\{S_h\}_h$ *provides order of approximation* k *in* L_p, $1 < p < \infty$. *That is, there exists a constant such that for any* $f \in \mathscr{H}^k(L_p(\mathbb{R}^d))$

$$E_p(f, S_h^h) \leqslant \text{const } h^k \, \|f\|_{\mathscr{H}_p{}^k}.$$

We note once more that the constant that we derive depends on p and blows us as p approaches either 1 or ∞. We recall also that the above theorem holds for $p = \infty$ as well, for a smaller than H^k_∞, class of smooth functions (see [3]).

Recently it was shown by Johnson [5] that S does not provide density order $k(\Xi)$, in other words, $k(\Xi)$ is an upper bound for the approximation order of S as well. Thus, we arrive at the following theorem:

THEOREM 4.2. *Let* M *be an exponential box spline that satisfies the regularity condition* (4.1). *Then, the approximation order of the family* $\{S_h\}_{h>0}$, *generated by* M *is* $k(\Xi)$.

It is worth pointing out that the finiteness of the right hand side of (4.1) is easily verifiable; as it was shown in [10] it is sufficient to assume that $\widehat{M_0} \in L_1(\mathbb{R}^d)$ with $M_0 := M_{\Xi, 0}$.

COROLLARY 4.3. *If* $\widehat{M_0} \in L_1(\mathbb{R}^d)$ *the order of approximation of the shift-invariant spaces generated by* M *is* $k(\Xi)$.

5. Proofs

For the proof of Theorem 4.1 we intend to test the assumptions of Theorem 2.3. It is easily seen that for sufficiently small $\varepsilon > 0$ and $h \in (0..1]$

$$\widehat{M_h}(\omega) > 1/2, \qquad \omega \in \mathscr{B}_\varepsilon.$$

Therefore for

$$m_{h,k,p}(\beta) := \left\| \frac{\eta_\varepsilon(\cdot)\, \widehat{M_h}(\cdot+\beta)}{(h^2+|\cdot|^2)^{k/2}\, \widehat{M_h}(\cdot)} \right\|_{\mathcal{M}_p} \tag{5.1}$$

we need onnly to prove that

$$\sup_{0<h<h_0} \|m_{h,k,p}\|_{l_1(2\pi\mathbb{Z}^d\setminus 0)} < \infty,$$

for some $h_0 > 0$ (cf. (2.13)).

PROPOSITION 5.1. *In the notations of Theorem* 4.1 *and* (5.1) *there exist* ε, $h_0 > 0$ *sufficiently small such that for every* $0 < h < h_0$ *and* $\beta \in 2\pi\mathbb{Z}^d\setminus 0$

$$m_{h,k,p}(\beta) < \text{const} \prod_{\xi \in K_\beta \cup L_\beta} |\xi\cdot\beta|^{-1}, \tag{5.2}$$

with the constant independent of β, h.

By virtue of Theorem 2.1, we have that for any integer $N > d/2$

$$m_{h,k,p}(\beta) \leqslant \text{const} \left\| \frac{\eta_\varepsilon(\cdot)\, \widehat{M_h}(\cdot+\beta)}{(h^2+|\cdot|^2)^{k/2}\, \widehat{M_h}(\cdot)} \right\|_N. \tag{5.3}$$

Note that

$$\frac{\widehat{M_h}(\omega+\beta)}{\widehat{M_h}(\omega)} = \prod_{\xi\in\Xi} \left(\frac{e^{h\lambda_\xi - i\xi\cdot(\omega+\beta)} - 1}{h\lambda_\xi - i\xi\cdot(\omega+\beta)} \right) \left(\frac{h\lambda_\xi - i\xi\cdot\omega}{e^{h\lambda_\xi - i\xi\cdot\omega} - 1} \right)$$

which after simplifications reduces to

$$\prod_{\xi\in L_\beta} \left(\frac{e^{h\lambda_\xi - i\xi\cdot(\omega+\beta)} - 1}{h\lambda_\xi - i\xi\cdot(\omega+\beta)} \right) \left(\frac{h\lambda_\xi - i\xi\cdot\omega}{e^{h\lambda_\xi - i\xi\cdot\omega} - 1} \right) \prod_{\xi\in K_\beta} \left(\frac{h\lambda_\xi - i\xi\cdot\omega}{h\lambda_\xi - i\xi\cdot(\omega+\beta)} \right).$$

Since for every $\beta \in 2\pi\mathbb{Z}^d$, $\# K_\beta \geqslant k(\Xi)$ $(k(\Xi) = \min\{\# K_\beta : \beta \in 2\pi\mathbb{Z}^d\})$, we can further split K_β into two disjoint sets $K_{\beta,k}$ and $K'_{\beta,k}$ such that

$$K_\beta = K_{\beta,k} \cup K'_{\beta,k},$$

with $K_{\beta,k}$, consisted of any $k := k(\Xi)$ vectors from K_β. In order to obtain an upper bound for the right hand side of (5.3) we write

$$\frac{\eta_\varepsilon(\omega)\, \widehat{M_h}(\omega+\beta)}{(h^2+|\omega|^2)^{k/2}\, \widehat{M_h}(\omega)} = \Lambda_h(\omega)\, \Pi_h(\omega)\, \Theta_h(\omega) \tag{5.4}$$

where

$$\Lambda_h(\omega) := \eta_{2\varepsilon}(\omega) \prod_{\xi \in K_{\beta,k}} \frac{h\lambda_\xi - i\xi\cdot\omega}{(h^2 + |\omega|^2)^{1/2}\,(h\lambda_\xi - i\xi\cdot(\omega+\beta))}$$

$$\Pi_h(\omega) := \eta_{2\varepsilon}(\omega) \prod_{\xi \in K'_{\beta,k}} \frac{h\lambda_\xi - i\xi\cdot\omega}{(h\lambda_\xi - i\xi\cdot(\omega+\beta))}$$

and

$$\Theta_h(\omega) := \eta_\varepsilon(\omega) \prod_{\xi \in L_\beta} \left(\frac{e^{h\lambda_\xi} - i\xi\cdot(\omega+\beta) - 1}{h\lambda_\xi - i\xi\cdot(\omega+\beta)}\right)\left(\frac{h\lambda_\xi - i\xi\cdot\omega}{e^{h\lambda_\xi - i\xi\cdot\omega} - 1}\right).$$

LEMMA 5.2. *Let $N > d/2$. Then, there exist ε, $h_0 > 0$ such that for every multi-intiger α, $|\alpha| \leqslant N$*

$$\sup_{0<h<h_0} \sup_{\omega \in \mathbb{R}^d} |\omega|^{|\alpha|}\,|D^\alpha \Theta_h(\omega)| \leqslant \text{const} \prod_{\xi \in L_\beta} |\xi\cdot\beta|^{-1}, \tag{5.5}$$

for some constant depending on Ξ.

Proof. From Leibniz's formula for $\omega \in \mathscr{B}_\varepsilon$ and $\alpha \neq 0$, we have

$$D^\alpha \Theta_h(\omega) = \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} D^{\alpha-\gamma}\left(\eta_\varepsilon(\omega) \prod_{\xi \in L_\beta} \frac{h\lambda_\xi - i\xi\cdot\omega}{e^{h\lambda_\xi - i\xi\cdot\omega} - 1}\right)$$
$$\times D^\gamma\left(\prod_{\xi \in L_\beta} \frac{e^{h\lambda_\xi - i\xi\cdot(\omega+\beta)} - 1}{h\lambda_\xi - i\xi\cdot(\omega+\beta)}\right). \tag{5.6}$$

Let now $\gamma \leqslant \alpha$. It is easily seen by differentiation that for sufficiently small ε, $h > 0$, say $\varepsilon < \varepsilon_1$ and $h < h_1$,

$$\left\| D^{\alpha-\gamma}\left(\eta_\varepsilon(\omega) \prod_{\xi \in L_\beta} \frac{h\lambda_\xi - i\xi\cdot\omega}{e^{h\lambda_\xi - i\xi\cdot\omega} - 1}\right)\right\|_{L_\infty(\mathscr{B}_\varepsilon)} < \text{const}, \tag{5.7}$$

for some constant independent of h and γ. Therefore, it suffices to prove that for some $h_2 > 0$

$$\sup_{0<h<h_2} \left\| D^\gamma\left(\prod_{\xi \in L_\beta} \frac{e^{h\lambda_\xi - i\xi\cdot(\omega+\beta)} - 1}{h\lambda_\xi - i\xi\cdot(\omega+\beta)}\right)\right\|_{L_\infty(\mathscr{B}_\varepsilon)} \leqslant \text{const} \prod_{\xi \in L_\beta} |\xi\cdot\beta|^{-1}, \tag{5.8}$$

Without any loss of generality we fix a column vector $\xi \in L_\beta$ and we consider only

$$D^\gamma\left(\frac{e^{h\lambda_\xi - i\xi\cdot(\omega+\beta)} - 1}{h\lambda_\xi - i\xi\cdot(\omega+\beta)}\right).$$

Differentiation shows that there exists a constant independent of $0<h\leqslant 1$ and β such that

$$\left|D^{\gamma}\left(\frac{e^{h\lambda_{\xi}-i\xi\cdot(\omega+\beta)}-1}{h\lambda_{\xi}-i\xi\cdot(\omega+\beta)}\right)\right|\leqslant \text{const} \sum_{1\leqslant\mu\leqslant|\gamma|+1} |h\lambda_{\xi}-i\xi\cdot(\omega+\beta)|^{-\mu}. \quad (5.9)$$

Following Ron [10] we note that the rationality of the matrix Ξ guarantees the existence of an integer m such that $m\Xi\in\mathbb{Z}^{d\times n}$. Therefore, $m\,|\xi\cdot\beta|\geqslant 2\pi$ for every $\xi\in L_{\beta}$, which implies that for sufficiently small h and ω, say $0<h<h_2$ and $\omega\in\mathscr{B}_{\varepsilon_2}$

$$|h\lambda_{\xi}-i\xi\cdot(\omega+\beta)|\geqslant\frac{|\xi\cdot\beta|}{2}.$$

It follows that the right hand side of (5.9) is

$$\begin{aligned}&\leqslant \text{const} \sum_{1\leqslant\mu\leqslant|\gamma|+1} |\xi\cdot\beta|^{-\mu}\\ &\leqslant \text{const}\ |\xi\cdot\beta|^{-1}.\end{aligned} \quad (5.10)$$

Retrospectively, we see that we should choose h_0 and ε such that $h_0<\min\{h_1, h_2, 1\}$ and $\varepsilon<\min\{\varepsilon_1, \varepsilon_2\}$. ▮

LEMMA 5.3. *Let $N>d/2$ and α, $|\alpha|\leqslant N$. In the notations of Lemma* 5.2

$$\sup_{0<h<h_0}\ \sup_{\omega\in\mathbb{R}^d} |\omega|^{|\alpha|}\,|{}^{\alpha}\Pi_h(\omega)|\leqslant \text{const} \prod_{\xi\in K'_{\beta,k}} |\xi\cdot\beta|^{-1}, \quad (5.11)$$

for some constant independent of $\beta\in 2\pi\mathbb{Z}^d\backslash 0$.

Proof. The proof follows easily by differentiation and the arguments in the proof of the previous lemma. ▮

LEMMA 5.4. *Let $N>d/2$ and α, $|\alpha|\leqslant N$. In the notations of Lemma* 5.2

$$\sup_{0<h<h_0}\ \sup_{\omega\in\mathbb{R}^d} |\omega|^{|\alpha|}\,|D^{\alpha}\Lambda_h(\omega)|\leqslant \text{const} \prod_{\xi\in K_{\beta,k}} |\xi\cdot\beta|^{-1}. \quad (5.12)$$

Proof. We'll consider each term of Λ_h separately. For this we fix $\xi\in K_{\beta,k}$ and we let $|\gamma|\leqslant|\alpha|$, and $\omega\in\mathscr{B}_{2\varepsilon}$. We claim that

$$|\omega|^{|\gamma|}\left|D^{\gamma}\left(\frac{h\lambda_{\xi}-i\xi\cdot\omega}{(h^2+|\omega|^2)^{1/2}}\,(h\lambda_{\xi}-i\xi\cdot(\omega+\beta))^{-1}\right)\right|\leqslant \text{const}\ |\xi\cdot\beta|^{-1}. \quad (5.13)$$

Indeed, it follows by differentiation that for $|\gamma| \leqslant |\alpha|$

$$\begin{aligned}
&|\omega|^{|\gamma|} \left| D^{\gamma} \left(\frac{h\lambda_{\xi} - i\xi \cdot \omega}{(h^2 + |\omega|^2)^{1/2}} \right) \right| \\
&\quad = |\omega|^{|\gamma|} \left| \sum_{0 \leqslant |\delta| \leqslant 1} \binom{\gamma}{\delta} D^{\delta}(h\lambda_{\xi} - i\xi \cdot \omega)\, D^{\gamma-\delta}((h^2 + |\omega|^2)^{-(1/2)}) \right| \\
&\quad \leqslant \text{const } |\omega|^{|\gamma|} \sum_{0 \leqslant |\delta| \leqslant 1} |D^{\delta}(h\lambda_{\xi} - i\xi \cdot \omega)|\, (h^2 + |\omega|^2)^{-(|\gamma| - |\delta| + 1)/2} \\
&\quad \leqslant \text{const}.
\end{aligned}$$

Taking into account our discussion in Lemma 5.2, and assuming that ε, h_0 are sufficiently small (5.13) follows by an application of Leibniz's formula.

At last (5.12) is derived by differentiation. ∎

Proof of Proposition 5.1. It is easily seen from (2.5) that the operator norm of the right hand side of (5.4) satisfies

$$\|\Lambda_h \Pi_h \Theta_h\|_{\mathscr{M}_p} \leqslant \text{const } \|\Lambda_h\|_N \|\Pi_h\|_N \|\Theta_h\|_N. \tag{5.14}$$

Putting together the three previous lemmas we see that for every $0 < h < h_0$ and $\beta \in 2\pi\mathbb{Z}^d \backslash 0$

$$m_{h,k,p}(\beta) \leqslant \text{const} \prod_{\xi \in K_\beta \cup L_\beta} |\xi - \beta|^{-1}. \quad ∎ \tag{5.15}$$

Proof of Theorem 4.1. We have to show that

$$\sup_{0 < h < h_0} \|m_{h,k,p}\|_{l_1(2\pi\mathbb{Z}^d \backslash 0)} < \infty.$$

As a consequence of Proposition 5.1 for every $0 < h < h_0$

$$\|m_{h,k,p}\|_{l_1(2\pi\mathbb{Z}^d \backslash 0)} \leqslant \text{const} \sum_{\beta \in 2\pi\mathbb{Z}^d \backslash 0} \prod_{\xi \in K_\beta \cup L_\beta} |\xi \cdot \beta|^{-1} < \infty \tag{5.16}$$

which proves the result. ∎

Acknowledgments

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